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CIRCLE GRIDS AND BIPARTITE GRAPHS OF DISTANCES

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For t fixed, $n+t$ points $A_1,A_2,\dots,A_n$ and $B_1,B_2,\dots,B_t$ are constructed in the plane with $O(\sqrt{n})$ distinct distances $d(A_iB_j)$. As a by-product we show that the graph of the k largest distances can contain a complete subgraph $K_{t,n}$ with $n=\Theta(k^2)$, which settles a problem of Erdős, Lovász and Vesztergombi.

1. Introduction

Given any $n+t$ points $\mathscr{A}=\{A_1,A_2,\dots,A_n\}$ and $\mathscr{B}=\{B_1,B_2,\dots,B_t\}$ of the plane denote by $F(\mathscr{A},\mathscr{B})$ the number of distinct distances $d(A_i,B_j)$ (for $i\le n$ and $j\le t$). For each fixed t, put

$$f_t(n)=\min\{F(\mathscr{A},\mathscr{B})\ ;\ |\mathscr{A}|=n,|\mathscr{B}|=t\}.$$

We want to determine the order of magnitude of $f_t(n)$ as a function of n while t, as mentioned above, is fixed.

The case $t=1$ lacks any interest at all, since $f_1(n)=1$ for all n. (Just put all the A_i on a circle around $B=B_1$.)

Proposition. *For all* $t\ge 2$,

$$f_t(n)\ge\left\lceil\sqrt{n/2}\right\rceil$$

and equality holds e.g. for $t=2$.

Proof. Obvious since each A_i must be one of the intersection points of two circles; one from a set of $f_t(n)$ circles about B_1 and another one, from a set of the same size, about B_2. ∎

One might guess that for $t\ge 3$ such a low order of magnitude cannot be attained; $f_t(n)$ will be much bigger than $c\sqrt{n}$.

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However, this is not the case. We are going to construct point-sets of $n+t$ points for every t and $n \geq 4t^3$ with $F(\mathcal{A},\mathcal{B}) \leq \sqrt{nt}+3t/2$ (see Theorem 2 below).

Another related problem is the following (see [4]). Let S be a set of points in the plane. Let us denote by $d_1 > d_2 > \dots$ the different distances determined by these points and by $G(S,k)$ the graph on the vertex set S obtained by joining two points iff their distance is at least d_k. Erdős, Lovász and Vesztergombi call $G(S,k)$ *"the graph of the k largest distances"* of S. They also posed the following problem there:

Given $t \geq 3$ and k, how large a complete bipartite graph $K_{t,n}$ can be contained in $G(S,k)$?

For any positive integers t and k put

$$g_t(k) = \max\{n \ ; \ \exists S \ \exists K_{t,n} \subset G(S,k)\}.$$

Again, the case $t=1$ is not worth mentioning. Otherwise, for $t \geq 2$, we have

$$g_t(k) = O(k^2)$$

e.g. from the previous Proposition. Also this order of magnitude will be shown to be best possible (in Theorem 3).

Our basic tools will be certain "grid-like" structures that we call *circle grids* which, together with our fundamental Theorem 1, are introduced in the next section. To give the reader a taste of their flavor, we briefly sketch here how to construct point sets that demonstrate

$$f_3(n) \leq c\sqrt{n}.$$

If we fix $\mathcal{B} = \{B_{-1}(-1,0); B_0(0,0); B_1(1,0)\}$ then for every A_i the distances between A_i and the B_j satisfy

$$d^2(A_i, B_0) = \frac{d^2(A_i, B_{-1}) + d^2(A_i, B_1)}{2} - 1.$$

This suggests that we make the squares of all distances (on the right hand side) to be $c\sqrt{n}$ consecutive members of the same arithmetic progression of integer terms; this guarantees that there will be at most $2c\sqrt{n}$ possible values for the left hand side. A bound of $f_3(n) \leq \sqrt{2n}$ can be proven this way. Circle grids will arise as generalizations of this simple idea.

If we want to decrease this coefficient $\sqrt{2}$ of $\sqrt{n}$, we can e.g. consider only those points for which the right hand side is an integer. This reduces the size of $\mathcal{A}$, as well as the number of the distinct $d(A_i, B_0)$, by a factor of 2 and results in a construction with $\sim \sqrt{n}$ different distances.

Remark. It is worth mentioning that we have been unable to further decrease the gap between the upper estimate $\sqrt{n}$ and the lower bound $\sqrt{n/2}$ of $f_3(n)$.

2. Results and open problems

Circle grids

In what follows straight lines are considered as degenerate circles.

Definition. A *circle grid* $\mathcal{G}$ of size $M \times N$ is a triple $\langle \mathcal{V}, \mathcal{H}, \mathcal{P} \rangle$ where the first two symbols,

$$\mathcal{V} = \{V_i; i = 1, 2, \ldots, M\} \quad \text{and}$$
$$\mathcal{H} = \{H_j; j = 1, 2, \ldots, N\}$$

denote sets of circles (or lines) and we require that each V_i intersects all the H_j. (We might call the curves in $\mathcal{V}$ and $\mathcal{H}$ "vertical" and "horizontal", respectively.) The last member of the triple,

$$\mathcal{P} = \{P_{ij}; i = 1 \ldots M, j = 1 \ldots N\}$$

is the *point set* of the grid where P_{ij} is a common point of V_i and H_j for $i = 1 \ldots M$, $j = 1 \ldots N$. (Note that by this definition different point-sets can correspond to the same sets of curves.)

Definition. Let s be a rational number. A subset $\mathcal{D}_s$ of $\mathcal{P}$ is a *diagonal set* of slope s if, for any $P_{i_1 j_1}, P_{i_2 j_2} \in \mathcal{D}_s$,

$$(j_1 - j_2) = s(i_1 - i_2).$$

$\mathcal{D}_s$ is *maximal* if it is not a proper subset of another diagonal set of the same slope (see some diagonal sets of slope 1 in Figure 1.).

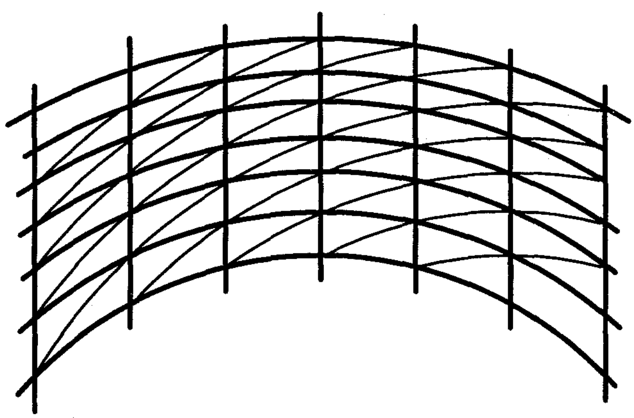

Fig. 1

The Theorems

Our aim is to construct circle grids with diagonals as "nice" as those of a usual rectangular grid.

Theorem 1. *For arbitrary integers M, N and $T \geq M^2$ there exists a circle grid $\mathcal{G}^T_{M,N}$ of size $(2M+1)\times N$ in the upper half-plane with the following properties:*

(i) *for any rational number s, each diagonal set $\mathcal{D}_s$ of slope s lies on a circle about $O_s=(s,0)$;*

(ii) *if ρ_s is the radius of this circle then*

$$s^2+T+1-sM \leq \rho_s^2 \leq s^2+T+N+sM;$$

(iii) *if, moreover, s is an integer then also ρ_s^2 is an integer;*

(iv) *the distances $d(P_{uv},P_{u'v'})$ are bounded by a quantity independent of T.*

[For large values of s, (i) is meaningless as all the D_s become singletons; however, e.g. for small integer values of s, it is rather strong a condition.]

It is worth to note that if we pick t consecutive integers as a range for s, then there will be at most $N+tM$ different radii $d(P_{u,v},O_s)$ (see Lemma 4). If, moreover, we make N and M proportional to $\sqrt{n}$, then we immediately get a bipartite graph with "few" distances. More detailed computations will prove the following assertions.

Theorem 2. *For $t\geq 3$ and $n\geq 4t^3$,*

$$f_t(n) \leq \sqrt{tn}+\frac{3}{2}t.$$

Remark. For certain values of t and n, also a better bound can be found, see Lemma 5. On the other hand side, if we need an estimate without the assumption on n, it is possible to find one that involves a term t^2 as well.

Also the Erdős–Lovász–Vesztergombi problem can be settled, using Theorem 1.

Theorem 3. *For $t\geq 3$ and $k\geq 2t^2$,*

$$g_t(k) \geq \frac{(k-2t)^2}{2t}.$$

[Again, for special values of t and k, better bounds exist, see Lemma 6.]

Unsolved problems

Several questions concerning circle grids remain open. First of all, the point sets (mentioned in Theorem 1) that we construct will have the degeneracy that all the B_j lie on a straight line.

Problem 1. Does Theorem 1 hold true even with the additional assumption that no three of the B_j be collinear?

The answer is unknown even for $t = 3$. It is not unlikely that solving this problem would require some deep number–theoretic insights.

Problem 2. Does $\lim_{n\to\infty} \frac{f_t(n)}{\sqrt{n}}$ and/or $\lim_{k\to\infty} \frac{g_t(k)}{k^2}$ exist and if so, determine them as a function of t.

The following problem of Erdős [1] was partially solved in [2]:

Are there n points $\{P_i; i=1\dots n\}$ in the plane that determine cn^2 unit circles, i.e for which there exist cn^2 different unit circles that contain three (or more) of the P_i each?

An affirmative answer to the above question would be implied by the following one (though we believe that the answer is negative):

Problem 3. Does there exist an $n \times n$ circle grid with all the V_i and the H_j unit circles, all whose diagonal sets of slope 1 also lie on unit circles?

A related question that also involves unit circles was formulated by L. A. Székely [3], who was looking for three sets that consist of n concurrent unit circles each and cover cn^2 points three times.

Yet another problem on other types of circle grids:

Problem 4. Does there exist a circle grid whose V_i and H_j as well as the diagonals D_s are all concentric circles?

[In the structure that we construct only the V_i lack this property.]

3. Proofs

Proof of Theorem 1.

Let N, M and $T \geq M^2$ be given. Define $\mathcal{V}$ (the "vertical" curves) to be the set of the vertical lines

$$L_u: \quad x = u/2 \qquad \text{for } u = -M, \dots, -1, 0, 1, \dots, M.$$

Let $\mathcal{H}$ (the "horizontal" curves) be the set of circles about the origin

$$C_v: \quad x^2 + y^2 = T + v \qquad \text{for } v = 1, 2, \dots, N.$$

Each of these C_v will intersect all the L_u since their radii $\sqrt{T+v}$ are larger than M by the assumption $T \geq M^2$.

Finally, we define the P_{uv} as those intersection points of the above lines and circles which are in the upper half-plane.

To prove that $O_s=(s;0)$ satisfies (i), we first mention that the coordinates of the P_{uv} are $(u/2;\sqrt{T+v-u^2/4})$ whence

$$d^2(P_{uv},O_s)=(u/2-s)^2+(T+v-u^2/4)=(s^2+T)+(v-su). \tag{1}$$

Here the first term of the last expression is a constant if s is fixed and the second one, while P_{uv} ranges over the points of a diagonal set of slope s, must also remain a constant.

To show (ii) and (iii), just observe that by (1),

$$\rho_s^2=d^2(P_{uv},O_s)=(s^2+T)+(v-su),$$

whence (iii) is obvious. The right hand side attains its minimum for $u=M$, $v=1$ and its maximum for $u=-M$, $v=N$, which proves (ii).

Also (iv) is clear from

$$\begin{aligned} d^2(P_{uv},P_{u'v'}) &= \left(\frac{u}{2}-\frac{u'}{2}\right)^2+\left(\sqrt{T+v-\frac{u^2}{4}}-\sqrt{T+v'-\frac{u'^2}{4}}\right)^2 \le \\ &\le \left(\frac{u}{2}-\frac{u'}{2}\right)^2+\left(T+v-\frac{u^2}{4}\right)-\left(T+v'-\frac{u'^2}{4}\right) \end{aligned}$$

where the two T's cancel. ∎

Before turning our attention to the proofs of the other two theorems, we estimate the number of distinct distances in the grid just constructed.

Lemma 4. *Assume $M\ge t-1$ and pick $\mathcal{B}=\left\{O_s\ ;\ s=0,\pm1,\pm2,\dots,\pm\frac{t-1}{2}\right\}$. Then the number of the distinct distances $d(P_{uv},O_s)$ is at most $N+(t-1)M$.*

Proof. On the one hand

$$d^2(P_{uv},O_s)=\rho_s^2\le s^2+T+N+sM\le\left(\frac{t-1}{2}\right)^2+T+N+\frac{t-1}{2}M$$

while on the other hand

$$\begin{aligned} d^2(P_{uv},O_s)=\rho_s^2 &\ge s^2+T+1-sM= \\ &=s(s-M)+T+1\ge \quad (\text{by } M\ge t-1) \\ &\ge\frac{t-1}{2}\left(\frac{t-1}{2}-M\right)+T+1=\left(\frac{t-1}{2}\right)^2+T+1-\frac{t-1}{2}M \end{aligned}$$

and, of course, there are $N+(t-1)M$ integers within this range. ∎

Proof of Theorem 2

First we prove a sharper bound for certain special values of t and n.

Lemma 5. *If $t\geq 3$ is odd, $n\geq 4t^3$ and $n=(t-1)(2M+1)^2$ for some M, then*

$$f_t(n) \leq \sqrt{n(t-1)} - \frac{t-1}{2}.$$

Proof. Put

$$N = \frac{t-1}{2}(2M+1)$$

and construct a circle grid with point set $\mathcal{P}$ in the upper half-plane with parameters M, N and $T\geq M^2$ arbitrary. [We do not need (iv) of Theorem 1 here.] Reflect $\mathcal{P}$ about the x–axis and let their union be $\mathcal{A}$. Then

$$|\mathcal{A}| = 2|\mathcal{P}| = 2N(2M+1) = (t-1)(2M+1)^2 = n.$$

Define, moreover,

$$\mathcal{B} = \left\{O_s \ ; \ s = 0, \pm 1, \pm 2, \ldots, \pm\frac{t-1}{2}\right\},$$

which makes $|\mathcal{B}|=t$.

We are left to calculate the number of distinct distances $d(P_{uv},O_s)$. However, by Lemma 4,

$$F(\mathcal{A},\mathcal{B}) \leq N + (t-1)M = \frac{t-1}{2}(2M+1) + (t-1)M =$$
$$= (t-1)(2M+1) - \frac{t-1}{2} = \sqrt{n(t-1)} - \frac{t-1}{2}. \quad \blacksquare$$

Now the general statement of Theorem 2 can be proven by substituting $t+1$ for t if t is even and, moreover, finding an $n'\geq n$ of type $t(2M+1)$ in place of n. Thus it is easy to see that $\sqrt{tn}-\sqrt{tn'}<2t$ whence the required upper bound follows immediately. ∎

Proof of Theorem 3

Just as in the previous proof, we consider special values first.

Lemma 6. *If $t\geq 3$ is odd, $k\geq 2t^2$ and $k=(t-1)(2M+1)$ for some M, then*

$$g_t(k) \geq \frac{k^2}{2(t-1)}.$$

Proof. Put

$$N = \frac{k}{2} = \frac{t-1}{2}(2M+1)$$

and construct a circle grid with point set $\mathcal{P}$ in the upper half-plane with parameters M, N as above while $T \geq M^2$ should be made big enough so that each distance $d(P_{uv}, O_s)$ be longer than all the $d(P_{uv}, P_{u'v'})$. [Here we heavily rely upon part (iv) of Theorem 1.]

Let $\mathcal{A} = \mathcal{P}$ and, as before,

$$\mathcal{B} = \left\{ O_s \ ; \ s = 0, \pm 1, \pm 2, \ldots, \pm \frac{t-1}{2} \right\},$$

which, again, makes $|\mathcal{B}| = t$. Moreover,

$$|\mathcal{A}| = N(2M+1) = \frac{k^2}{2(t-1)}.$$

The point set we have been looking for is defined to be

$$S = \mathcal{A} \cup \mathcal{B}.$$

Thus the longest distances within S occur between $\mathcal{A}$ and $\mathcal{B}$ (by the choice of T). Hence in order to prove that $G(S,k)$ does indeed contain a $K_{t,n}$, it suffices to show that among the points of $\mathcal{A}$ and $\mathcal{B}$ there are at most k distinct distances $d(P_{uv}, O_s)$.

Again, by Lemma 4, the number of these distances is at most

$$N + (t-1)M < N + \frac{t-1}{2}(2M+1) = \frac{k}{2} + \frac{k}{2} = k. \qquad \blacksquare$$

Also the proof of Theorem 3 can be completed by plugging $t+1$ in t and picking an appropriate $k' \leq k$ of the desired type. $\blacksquare$

References

[1] P. ERDŐS: Some applications of graph theory and combinatorial methods to number theory and geometry, *Algebraic methods in Graph Theory, Coll. Math. Soc. J. Bolyai* **25**, (1981), 137–148.

[2] G. ELEKES: n points in the plane can determine $n^{3/2}$ unit circles, *Combinatorica* **4** (2–3) (1984) 131.

[3] L. A. SZÉKELY: Private communication.

[4] P. ERDŐS, L. LOVÁSZ, K. VESZTERGOMBI: On graphs of large distances, *Discrete and Computational Geometry*, 4 (1989), 541–549.

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